

- Telephium repens folio non deciduo. C. B. P.
287.
- 1646 Thalictrum minus alterum Parisiense foliis
crassioribus et lucidis. H. R. Par.
- 1647 Tithymalus maritimus. C. B. P. 291.
Tithimalus paralius. J. B. 3. 674.
- 1648 Trifolium Bitumen redolens. C. B. 327.
- 1649 Trifolium Bitumen redolens angustifolium.
Boer. Ind. Alt. 2. p. 32.
Trifolium bituminosum arboreum angustifo-
lium ac sempervirens. Hort. Cath.
- 1650 Virga aurea Canadensis foliis carnosis non ser-
ratis latioribus. Hist. Oxon.

XIX. *A Letter to the Right Honourable
George Earl of Macclesfield, President of
the Royal Society, on the Advantage of
taking the Mean of a Number of Obser-
vations, in practical Astronomy: By
T. Simpson, F. R. S.*

My Lord,

Read April 10, 1755. **I**T is well known to your Lordship,
that the method practised by astron-
omers, in order to diminish the errors arising from the
imperfections of instruments, and of the organs of sense,
by taking the Mean of several observations, has not
been so generally received, but that some persons, of
considerable note, have been of opinion, and even
publicly maintained, that one single observation,
taken

taken with due care, was as much to be relied on as the Mean of a great number.

As this appeared to me to be a matter of much importance, I had a strong inclination to try whether, by the application of mathematical principles, it might not receive some new light; from whence the utility and advantage of the method in practice might appear with a greater degree of evidence. In the prosecution of this design (the result of which I have now the honour to transmit to your Lordship) I have, indeed, been obliged to make use of an hypothesis, or to assume a series of numbers, to express the respective chances for the different errors to which any single observation is subject; which series, to me, seems not ill-adapted: but this I shall submit intirely to the judgment of your Lordship, who have made so great a number of observations, at your seat at Shirburn; where, to the best collection of mathematical books, your Lordship has added a more complete set of astronomical instruments than (perhaps) are to be found in the possession of any nobleman in Europe.

Should not the assumption, which I have made use of, appear to your Lordship so well chosen as some others might be, it will, however, be sufficient to answer the intended purpose: and your Lordship will find, on calculation, that, whatever series is assumed for the chances of the happening of the different errors, the result will turn out greatly in favour of the method now practised, by taking a mean value. But I shall no longer detain your Lordship with general observations, but proceed to the

matter proposed; which I shall consider in the following propositions.

PROPOSITION I.

Supposing that the several chances for the different errors that any single observation can admit of, are expressed by the terms of the progression $r^{-v} \dots r^{-3}, r^{-2}, r^{-1}, r^0, r^1, r^2, r^3 \dots r^v$ (where the exponents denote the quantities and qualities of the particular errors, and the terms themselves the respective chances for their happening): 'tis proposed to determine the probability, or odds, that the error, by taking the Mean of a given number (n) of observations, exceeds not a given quantity $\left(\frac{m}{n}\right)$.

It is evident, from the laws of chance, that, if the given series, $r^{-v} \dots + r^{-3} + r^{-2} + r^{-1} + r^0 + r^1 + r^2 + r^3 \dots + r^v$, expressing all the chances in one observation, be raised to the n^{th} power, the terms of the series thence arising will truly exhibit all the different chances in all the proposed (n) observations. In order to raise this power, with the greatest facility, our given expression may be reduced

to $r^{-v} \times \frac{1 - r^{2v+1}}{1 - r}$: whereof the n^{th} power (making

$w = 2v + 1$), will be $r^{-nv} \times \overline{1 - r^w}^n \times \overline{1 - r}^{-n}$; which, expanded, becomes

$$r^{-nv} - nr^{w-nv} + \frac{n}{1} \cdot \frac{n-1}{2} r^{2w-nv} - \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} r^{3w-nv} + \&c.$$

multiplied into

$$1 + nr + \frac{n}{1} \cdot \frac{n+1}{2} r^2 + \frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} r^3 + \&c.$$

Now,

Now, to find from hence the sum of all the chances whereby the excess of the positive errors above the negative ones can amount, precisely, to a given number m , it will be sufficient (instead of multiplying the former series by the whole of the latter) to multiply by such terms of the latter, only, as are necessary to the production of the given exponent m , in question. Thus, the first term (r^{-nv}) of the former series is to be multiplied by that term of the second, whose exponent is $nv + m$, in order that the power of r , in the product, may be r^m . But it is plain, from the law of the series, that the coefficient of this term (putting $nv + m = q$) will be $\frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} (q)$, q being the number of factors; and consequently, that the product under consideration will be $\frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} (q) \times r^m$. Again, the second term of the former series being $-nr^{w-nv}$, the exponent of the corresponding term of the latter will be $-w + nv + m (= q - w)$, and therefore the term itself equal to $\frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} (q-w) \times r^{q-w}$: which, drawn into $-nr^{w-nv}$, gives $\frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} (q-w) \times -nr^m$, for the second term required.

In like manner, the third term, of the product, whose exponent is m , will be found $\frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} (q-2w) \times \frac{n}{1} \cdot \frac{n-1}{2} r^m$: and the sum of all the terms having the same given exponent (m) will consequently be

$$\begin{aligned}
 & + \frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} (q) \times r^m \\
 & - \frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} (q-w) \times n r^m \\
 & + \frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} (q-2w) \times \frac{n}{1} \cdot \frac{n-1}{2} r^m \\
 & - \frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} (q-3w) \times \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} r^m \\
 & \quad \mathcal{E}c. \quad \mathcal{E}c.
 \end{aligned}$$

From which general expreffion, by expounding m by 0, + 1, — 1, + 2, — 2, &c. fucceffively, the fum of all the chances, whereby the difference of the positive and negative errors can fall within the propofed limits, will be found; which, divided by $r^{-m} \times \overline{1-rw}^n \times \overline{1-r}^n$, will give the true meafure of the probability required: from whence the advantage of taking the Mean of feveral obfervations might be fhewn: but this I fhall exemplify in the next propofition; which is better adapted to the purpofe, and to which this is premifed, as a Lemma.

Remark.

If r be taken = 1, or the chances for the errors in excefs and defect be fupposed exactly the fame; then our expreffion, by expunging the powers of r , will become the very fame with that fhewing the chances for throwing $n+q$ points with n dice; each die having as many faces (w), as the refult of any one fingle obfervation, can come out different ways. Which may be otherwife made to appear, independent of any kind of calculation, from the bare confider-

consideration, that the chances for throwing precisely the number m , with n dice, whereof the faces, of each, are numbered $-v, -v-1, -v-2, \dots, -3, -2, -1, 0, +1, +2, +3, \dots, +v$, must be the very same as the chances whereby the positive errors can exceed the negative ones by that precise number; which last are, evidently, the same as the chances for throwing precisely the number $v+1.n+m$ (or $n+q$) with the same n dice, when they are numbered in the common way, with the terms of the natural progression $1, 2, 3, 4, 5, \&c.$: because the number upon each face being, here, increased by $v+1$, the whole increase upon all the n faces will be expressed by $v+1.n$; so that there will be now the very same chance for the number $v+1.n+m$, as there was before for the number m ; since the chances for throwing any faces assigned will continue the same, however those faces are numbered.

PROPOSITION II.

Supposing the respective chances, for the different errors which any single observation can admit of, to be expressed by the terms of the series $r^{-v} + 2r^{1-v} + 3r^{2-v} + \dots + v+1.r^0 + 3r^{v-2} + 2r^{v-1} + r^v$ (whereof the coefficients, from the middle one ($v+1$), decrease, both ways, according to the terms of an arithmetical progression): 'tis proposed to determine the probability, or odds, that the error, by taking the Mean of a given number (t) of observations, exceeds not a given quantity $\left(\frac{m}{t}\right)$.

Pursuing

Pursuing the method laid down in the preceding problem, the sum of the series here given will appear to be $r^{-v} \times \frac{1-r^{v+1}}{1-r}^2$ (being the same

with the square of the geometrical progression $r^{-\frac{1}{2}v} \times 1 + r + r^2 + r^3 \dots + r^v$), and the t^{th} power thereof, by making $n = 2t$, and $w = v + 1$. will therefore be given $= r^{-tv} \times \frac{1-r^w}{1-r}^n \times \frac{1-r}{1-r}^{-n} =$

$r^{-tv} - nr^{w-tv} + \frac{n}{1} \cdot \frac{n-1}{2} r^{2w-tv} + \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} r^{3w-tv} + \dots$
multiplied into

$$1 + nr + \frac{n}{1} \cdot \frac{n+1}{2} r^2 + \frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} r^3 + \dots$$

Which series being the same with those in the preceding problem (excepting only that the exponents of the former of them are expressed in terms of t , instead of n), it is plain, therefore, that if q be made $= tv + m$ (instead of $nv + m$), the conclusion, there brought out, will answer equally here: so that the sum of all the chances whereby the excess of the positive errors above the negative ones, can amount to a given number m , precisely, will be truly represented by

$$\begin{aligned} & + \frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} (q) \times r^m \\ & - \frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} (q-w) \times nr^m \\ & + \frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} (q-2w) \times \frac{n}{1} \cdot \frac{n-1}{2} r^m \\ & - \frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} (q-3w) \times \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} r^m \\ & \dots \end{aligned}$$

But

But this general expression, as several of the factors destroy each other, may be transformed to another, which is more commodious. Thus $\frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} (q)$, in the first line, will, by breaking the numerator and denominator into two parts, become

$$\frac{n \cdot \overline{n+1} \cdot \overline{n+2} \cdot \overline{n+3} \cdot \dots \cdot \overline{q \times q+1} \cdot \overline{q+2} \cdot \overline{q+3} \cdot \dots \cdot \overline{q+n-1}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot \dots \cdot n \cdot \overline{n+1} \cdot \overline{n+2} \cdot \overline{n+3} \cdot \dots \cdot q} :$$

which, by equal division, is reduced to

$$\frac{q \cdot \overline{q+1} \cdot \overline{q+2} \cdot \dots \cdot \overline{q+n-1}}{1 \cdot 2 \cdot 3 \cdot \dots \cdot n-1} = \frac{q+n-1 \cdot \overline{q+n-2} \cdot \dots \cdot q}{1 \cdot 2 \cdot \dots \cdot n-1} =$$

$$\frac{p-1}{1} \cdot \frac{p-2}{2} \cdot \frac{p-3}{3} (n-1) ; \text{ supposing } p (= q+n) = v + m + n.$$

In the very same manner, making $q' = q - w$, and $p' (= q' + n) = p - w$, it appears, that $\frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} (q-w) = \frac{p'-1}{1} \cdot \frac{p'-2}{2} \cdot \frac{p'-3}{3} (n-1)$, &c. &c.

and consequently, that our whole given expression (supposing $p'' = p - 2w$, $p''' = p - 3w$, &c.) will be transformed to

$$\begin{aligned} & + \frac{p-1}{1} \cdot \frac{p-2}{2} \cdot \frac{p-3}{3} (n-1) \times r^m \\ & - \frac{p'-1}{1} \cdot \frac{p'-2}{2} \cdot \frac{p'-3}{3} (n-1) \times n r^m \\ & + \frac{p''-1}{1} \cdot \frac{p''-2}{2} \cdot \frac{p''-3}{3} (n-1) \times \frac{n}{1} \cdot \frac{n-1}{2} r^m \\ & - \frac{p'''-1}{1} \cdot \frac{p'''-2}{2} \cdot \frac{p'''-3}{3} (n-1) \times \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} r^m \\ & \quad \quad \quad \text{\&c.} \quad \quad \quad \text{\&c.} \end{aligned}$$

Which expression is to be continued till the factors become nothing, or negative; and which, when

$r=1$, will be the very same with that exhibiting the number of chances for p points, precisely, with n dice, having each w faces: and in this case, where the chances for the errors in excess and defect are the same, the solution is the most simple it can be; since, from the chances above determined, answering to the number p , precisely, the sum of the chances for all the inferior numbers (inclusive) may be readily obtained; being given (from the Method of Increments) equal to

$$\begin{aligned}
 & + \frac{p}{1} \cdot \frac{p-1}{2} \cdot \frac{p-2}{3} \cdot \frac{p-3}{4} (n) \\
 & - \frac{p'}{1} \cdot \frac{p'-1}{2} \cdot \frac{p'-2}{3} \cdot \frac{p'-3}{4} (n) \times n \\
 & + \frac{p''}{1} \cdot \frac{p''-1}{2} \cdot \frac{p''-2}{3} \cdot \frac{p''-3}{4} (n) \times \frac{n}{1} \cdot \frac{n-1}{2} \\
 & - \frac{p'''}{1} \cdot \frac{p'''-1}{2} \cdot \frac{p'''-2}{3} \cdot \frac{p'''-3}{4} (n) \times \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \\
 & \quad \quad \quad \mathcal{E}c. \quad \quad \quad \mathcal{E}c.
 \end{aligned}$$

The difference between which and half (w^n), the sum of all the chances, (which difference I shall denote by D), will consequently be the number of the chances whereby the errors in excess (or in defect) can fall within the given limit m : so that $\frac{D}{\frac{1}{2}w^n}$ will be the true measure of the required probability, that the error, by taking the Mean of t observations, exceeds not the quantity $\frac{m}{t}$, proposed.

To illustrate this by an example, from whence the utility of the method in practice, may clearly appear, it will be necessary, in the first place, to assign some number for v , expressing the limits of
the

the errors to which any observation is subject. These limits, indeed, depend on the goodness of the instrument, and the skill of the observer; but I shall suppose here, that every observation may be relied on to 5 seconds; and that the chances for the several errors, $-5''$, $-4''$, $-3''$, $-2''$, $-1''$, $0''$, $+1''$, $+2''$, $+3''$, $+4''$, $+5''$, included within the limits thus assigned, are respectively proportional to the terms of the series 1, 2, 3, 4, 5, 6, 5, 4, 3, 2, 1: which series seems much better adapted than if all the terms were to be equal, since it is highly reasonable to suppose, that the chances for the different errors decrease, as the errors themselves increase.

These particulars being premised, let it be now required to find, what the probability, or chance, for an error of 1, 2, 3, 4, or 5 seconds will be, when (instead of relying on one) the Mean of six observations is taken. Here, then, v being $=5$, and $t=6$, we have $n (=2t) = 12$, $w (=v+1) = 6$, and $p (=tv+n+m) = 42+m$: but the value of m , if we first seek the chances whereby the error exceeds not 1 second, will be had from the equation $\frac{m}{t} = \pm 1$; where either

sign may be used, but the negative one is the most commodious: from whence we have $m (= -t) = -6$; and therefore $p=36$, $p'=30$, $p''=24$, $p'''=18$, &c. which values being substituted in the general expression above determined, it will become

$$\frac{36}{1} \cdot \frac{35}{2} \cdot \frac{34}{3} (12) - \frac{30}{1} \cdot \frac{29}{2} \cdot \frac{28}{3} (12) \times 12 + \frac{24}{1} \cdot \frac{23}{2} \cdot \frac{22}{3} (12) \times 66 - \frac{18}{1} \cdot \frac{17}{2} \cdot \frac{16}{3} (12) \times 220 = 299576368:$$

and this subtracted from 1088391168 ($= 1 \times 6^{12}$), leaves 788814800, for the value of D corresponding.

Therefore the required probability, that the error, by taking the Mean of six observations, exceeds not a single second, will be truly measured by the fraction $\frac{788814800}{1088391168}$; and consequently the odds will be as

788814800 to 299576368, or as $2\frac{2}{3}$ to 1, nearly.

But the proportion, or odds, when one single observation is relied on, is only as 16 to 20, or as $\frac{8}{5}$ to 1.— To find, now, the probability, that the re-

sult comes within 2 seconds of the truth, let $\frac{m}{t}$

be made = - 2; so shall $m (= - 2t) = - 12$; and

therefore $p=30$, $p'=24$, $p''=18$, &c. And our

general expression will here come out = 36079407;

and consequently $D = 1052311761$: whence

$\frac{1052311761}{1088391168}$ will be the true measure of the probabi-

lity here sought; and the odds, or proportion of the chances, will therefore be as 1052311761 to

36079407, or as 29 to 1, nearly. But the propor-

tion, or odds, when one single observation is relied on, is only as 2 to 1: so that the chance, for an

error exceeding 2 seconds, is not $\frac{1}{10}$ part so great

from the Mean of six, as from one single observa-

tion. And it will be found, in the same manner,

that the chance for an error exceeding 3 seconds, will

not be $\frac{1}{100}$ part so great from the Mean of six, as

from one single observation. Upon the whole of

of which it appears, that the taking of the Mean of

a number of observations, greatly diminishes the

chances for all the smaller errors, and cuts off al-

most all possibility of any great ones: which last

consideration, alone, seems sufficient to recommend

use of the method, not only to astronomers, but to all others concerned in making of experiments of any kind (to which the above reasoning is equally applicable). And the more observations or experiments there are made, the less will the conclusion be liable to err, provided they admit of being repeated under the same circumstances.

Other examples, and particulars might be added, in confirmation of what is here determined; but as I would not appear tedious to your Lordship, I here conclude, who am,

Woolwich, My Lord,
March 4, 1755.

Your Lordship's

most obedient humble servant,

T. Simpson.

XX. *An Account of the Success of Agaric,
and the Fungus vinosus, in Amputations:*
By Mr. James Ford, Surgeon, of Bristol.

Bristol, March 31, 1755.

Read April 10, 1755. **I**N 1753 I had some pieces of the agaric of the oak brought me from France, which I have frequently used with success in hæmorrhages, particularly once after the operation for the stone, where a large artery lay so deep, that it could not conveniently be taken up with a needle. After the publication of Mr. Warner's observations, Mr. Thornhill applied it successfully to
an